Applications of Partial Differential Equations

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Overview

Partial differential equations are classified in many ways. One such way is classifying them as linear and non-linear. Linear equations have an algebraic nature in their solution sets; in the sense that these solutions can be superimposed. Nonlinear equations do not share this property.

Equally important in classification schemes of a PDE is the specific nature of the physical phenomenon that it describes; for example, a PDE can be classified as wave-like, diffusion like, or static, depending upon whether it models wave propagation, a diffusion process, or an equilibrium state, respectively. For example, Laplace's equation is a linear equilibrium equation; the heat equation is a linear diffusion equation because the heat flow is a diffusion process. In three lectures, we discuss some physical examples and methods for solving them using PDE as a tool.

Classification of Partial Differential Equations of Second order

The general form of second order partial differential equation is given by

$$A\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial x \partial y} + C\frac{\partial^2 z}{\partial y^2} + D\frac{\partial z}{\partial x} + E\frac{\partial z}{\partial y} + Fz = G(x, y)$$

where A, B, C, D, E, F and G are functions of x and y or constants.

Now consider the term

$$B^2 - 4AC.$$

(i) If $B^2 - 4AC < 0$, then the equation is Elliptic

(ii) If $B^2 - 4AC > 0$, then the equation is Hyperbolic

(iii) If $B^2 - 4AC = 0$, then the equation is Parabolic.

Exercise 1.

Classify the following equations

1. $u_{xx} - 3u_{xy} + u_{yy} = 0$
   Solution. *Here* $A = 1$, $B = -3$ *and* $C = 1$. *Therefore*

$$B^2 - 4AC = (-3)^2 - 4(1)(1) = 9 - 4 = 5 > 0$$

   Hence the given equation is hyperbolic.

2. $4u_{xx} - 7u_{xy} + 3u_{yy} = 0$
   Solution. *Here* $A = 4$, $B = -7$ *and* $C = 3$. *Therefore*

$$B^2 - 4AC = (-7)^2 - 4(4)(3) = 49 - 48 = 1 > 0$$

   Hence the given equation is hyperbolic.

Exercise 2.

Classify the following equations

3. $c^2u_{xx} + 2cu_{xy} + u_{yy} = 0$, $a \neq 0$
   Solution. *Here* $A = c^2$, $B = 2c$ *and* $C = 1$. *Therefore*

$$B^2 - 4AC = (2c)^2 - 4(c^2)(1) = 4c^2 - 4c^2 = 0$$

   Hence the given equation is Parabolic.

4, $u_{xx} + 2u_{xy} + 5u_{yy} = 0$
   Solution. *Here* $A = 1$, $B = 2$ *and* $C = 5$. *Therefore*

$$B^2 - 4AC = (2)^2 - 4(1)(5) = 4 - 20 = -16 < 0$$

   Hence the given equation is elliptic.

Exercise 3.

Classify the following equations

5. $8u_{xx} - 2u_{xy} - 3u_{yy} = 0$
   Solution. *Here* $A = 8$, $B = -2$ *and* $C = -3$. *Therefore*

$$B^2 - 4AC = (-2)^2 - 4(8)(-3) = 4 + 96 = 100 > 0$$

   Hence the given equation is hyperbolic.

6. $f_{xx} + k^2 f_{yy} = 0$
   Solution. *Here* $A = 1$, $B = 0$ *and* $C = k^2$. *Therefore*

$$B^2 - 4AC = 0 - 4(1)(k^2) = -4k^2$$

   If $k = 0$ *then*

$$B^2 - 4AC = 0,$$

   the given equation is Parabolic

Exercise 4.

Classify the following equations

7. $4\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} - 6\frac{\partial u}{\partial x} - 8\frac{\partial u}{\partial y} - 16u = 0$
   Solution. *Here $A = 4$, $B = 4$ and $C = 1$. Therefore*
   $$B^2 - 4AC = (4)^2 - 4(4)(1) = 16 - 16 = 0$$
   Hence the given equation is parabolic.
8. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$
   Solution. *Here $A = 1$, $B = 0$ and $C = 1$. Therefore*
   $$B^2 - 4AC = 0 - 4(4)(1) = -4 < 0$$
   Hence the given equation is elliptic.

Exercise 5.

Classify the following equations

9. $y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} + 2u_x - 3u_y = 0$
   Solution. *Here* $A = y^2$, $B = -2xy$ *and* $C = x^2$. *Therefore*

$$B^2 - 4AC = (-2xy)^2 - 4(y^2)(x^2) = 4x^2y^2 - 4x^2y^2 = 0$$

   Hence the given equation is parabolic.

10. $y^2u_{xx} + x^2u_{yy} + u_x^2 + u_y^2 + 7 = 0$
    Solution. *Here* $A = y^2$, $B = 0$ *and* $C = 1$. *Therefore*

$$B^2 - 4AC = 0 - 4(y^2)(1) = -4y^2$$

    If $y = 0$ *then* $B^2 - 4AC = 0$, *the given equation is parabolic.*
    If $y \neq 0$ *then* $B^2 - 4AC < 0$, *the given equation is elliptic.*

Exercise 6.

Classify the following equations

11. $u_{xx} + xu_{yy} = 0$
Solution. *Here* $A = 1$, $B = 0$ *and* $C = x$. *Therefore*

$$B^2 - 4AC = 0 - 4(1)(x) = -4x$$

If $x = 0$ *then* $B^2 - 4AC = 0$, *the given equation is parabolic.*
If $x > 0$ *then* $B^2 - 4AC < 0$, *the given equation is elliptic.*
If $x < 0$ *then* $B^2 - 4AC > 0$, *the given equation is hyperbolic.*

12. $x^2u_{xx} + 2xu_{xy} + (1 - y^2)u_{yy} = 0$
Solution. *Here* $A = x^2$, $B = 2x$ *and* $C = 1 - y^2$. *Therefore*

$$B^2 - 4AC = (2x)^2 - 4(x^2)(1 - y^2) = 4x^2y^2$$

If $x = 0, y = 0$ *then* $B^2 - 4AC = 0$, *the given equation is parabolic. If* $x \neq 0, y \neq 0$ *then* $B^2 - 4AC > 0$, *the given equation is hyperbolic.*

Method of Separation of Variables

This is the one of the simple and mostly used method to solve the second order partial differential equations. Since in all our partial differential equations we take z as a dependent variable and x and y as independent variables, then the relation $z = f(x, y)$ to be the solution.

In this method we assume that the solution is the product of two functions, one of them is function of x alone and the other a function of y alone. By this the partial differential equation now converted into ordinary differential equations and this equations can solved easily. The following examples illustrate us how this procedure works.

Example 7.

Solve $3x\frac{\partial z}{\partial x} - 2y\frac{\partial z}{\partial y} = 0$ *by the method of separation of variables.*

Solution. *The given differential equation is* $3x\frac{\partial z}{\partial x} - 2y\frac{\partial z}{\partial y} = 0$.
Let us assume that $z = X(x)Y(y)$ *be the solution of the above equation.*
Then

$$\frac{\partial z}{\partial x} = X'Y \quad \text{and} \frac{\partial z}{\partial y} = XY'.$$

Substituting these values in partial differential equation we get $3xX'Y - 2yXY' = 0$.
Hence we get $3x\frac{X'}{X} = 2y\frac{Y'}{Y} = k$ *(say). Now* $3x\frac{X'}{X} = k$ *gives* $\frac{X'}{X} = \frac{k}{3x}$, *and hence* $X = c_1 x^{\frac{k}{3}}$.
In a similar way $2y\frac{Y'}{Y} = k$ *gives* $Y = c_2 y^{\frac{k}{2}}$. *Hence* $z = c_1 x^{\frac{k}{3}} c_2 y^{\frac{k}{2}} = cx^{\frac{k}{3}} y^{\frac{k}{2}}$, *where* $c = c_1 c_2$.

Example 8.

Solve $2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$, $z(x,0) = 5e^{-x}$ *by the method of separation of variables.*

Solution. *The given differential equation is* $2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.
Let us assume that $z = X(x)Y(y)$ *be the solution of the above equation. Then* $\frac{\partial z}{\partial x} = X'Y$ *and* $\frac{\partial z}{\partial y} = XY'$. *Substituting these values in partial differential equation we get* $2X'Y + XY' = 0$ *and hence* $2\frac{X'}{X} = -\frac{Y'}{Y} = k$ *(say). Now,* $2\frac{X'}{X} = k$ *gives* $X = c_1 e^{\frac{k}{2}x}$.
In a similar way, $\frac{Y'}{Y} = k$ *gives* $Y = c_2 e^{ky}$. *Hence*

$$\begin{aligned} z(x,y) &= c_1 e^{\frac{k}{2}x} c_2 e^{ky} \\ &= c e^{\frac{k}{2}x} e^{ky} \end{aligned}$$

where $c = c_1 c_2$. *Give that* $z(x,0) = 5e^{-x}$. *So* $z(x,0) = ce^{\frac{k}{2}x}e^{k(0)}$ *and* $5e^{-x} = ce^{\frac{k}{2}x}$.
Solving them, we get $c = 5$ *and* $\frac{k}{2} = -1 \Rightarrow k = -2$.
Substituting these values we get

$$z(x,y) = 5e^{-(x+2y)}.$$

One dimensional Wave Equation

Let us consider a tightly stretched elastic string of length l is fixed at the end points 0 and A, subject to a constant tension T. The string is released from rest and allowed to vibrate. Now we shall determine the displacement $y(x,t)$ of the point x of the string at time $t > 0$. We make the following assumptions.

1. The mass per unit length of the string is constant.
2. The string is perfectly elastic and so it does not offer resistance to bending.
3. The tension caused by stretching the string before fixing it to the end points is constant at all the points of the deflected string at any time.
4. The tension T is so large such that the gravitational force and the friction may be neglected.
5. The string performs a small transverse motion in a vertical plane, that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string remain small in absolute value.

One dimensional Wave Equation

From these assumptions we may except that the solution $y(x, t)$ of the differential equation to be obtained will reasonably well to describe small vibrations of the string.
To derive the differential equation let us consider the forces acting on a small portion PQ where $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ of the string (see fig.)

One dimensional Wave Equation

Let ψ and $\psi + \Delta\psi$ be the angles made by the tangents at P and Q respectively with the $x - axis$ and $PQ = \Delta s$.

By Newton's second law of motion, the total force acting on this piece of string is equal to the mass of the string multiplied by its acceleration. That is,

$$\text{Force} = \text{mass} \times \text{acceleration} \tag{1}$$

$$= (m\Delta s)\frac{\delta^2 y}{\delta t^2} \tag{2}$$

Here to find acceleration we take partial derivative of y, w.r.t. t because y is a function of two variable. We assume in this equation that the string is moving only in the xy-plane and that each particle in the string moves only vertically. The actual external force acting on PQ in the positive y direction is

$$= T\sin(\psi + \Delta\psi) - T\sin\psi$$

$$= T(\psi + \Delta\psi - \psi) = T\Delta\psi \tag{3}$$

Equating (1) and (2) we have

$$(m\Delta s)\frac{\partial^2 y}{\partial t^2} = T\Delta\psi$$
$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\Delta\psi}{\Delta s}$$

Now taking the limit as $Q \longrightarrow P$ we get

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{d\psi}{ds} \tag{4}$$

where $\frac{d\psi}{ds}$ is the curvature at P is given by

$$\frac{d\psi}{ds} = \frac{1}{\rho} = \frac{\frac{\partial^2 y}{\partial x^2}}{\left[1+\left(\frac{\partial y}{\partial x}\right)^2\right]^{\frac{3}{2}}} = \frac{\partial^2 y}{\partial x^2}$$

since $\left(\frac{\partial y}{\partial x}\right)^2$ is very small by the assumption (5). Hence we get

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2} = a^2\frac{\partial^2 y}{\partial x^2}, \quad \text{where } a^2 = \frac{T}{m}. \tag{5}$$

Solution of wave equation by separation of variables

The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

We try to solve the equation by separation of variable method.Let

$$y(x,t) = X(x)T(t) \tag{2}$$

be the solution of the above equation with $X(x)$ is a function of x alone and T is a function of t alone.
Differentiating (2) partially with respect to x and t we get

$$\frac{\partial^2 y}{\partial x^2} = X''T \quad \text{and} \quad \frac{\partial^2 y}{\partial t^2} = XT''.$$

Solution of wave equation by separation of variables

Substituting these values in (1) we get

$$XT'' = a^2 X'' T$$

$$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} = k \qquad \text{say}$$

$$\frac{X''}{X} = k \qquad \text{and} \qquad \frac{T''}{a^2 T} = k$$

$$X'' - kX = 0 \tag{3}$$

and

$$T'' - ka^2 T = 0 \tag{4}$$

Case 1. If $k = 0$ then

$$X'' = 0 \qquad\qquad T'' = 0$$

$$\frac{d^2X}{dx^2} = 0 \qquad\qquad \frac{d^2T}{dt^2} = 0$$

$$X(x) = (c_1x + c_2) \qquad\qquad T(t) = (c_3t + c_4)$$

Solution of wave equation by separation of variables

Case 2. If k is positive then $k = p^2$ (say)

$$X'' - p^2 X = 0 \qquad T'' - p^2 a^2 t = 0$$

$$\frac{d^2X}{dx^2} - p^2 X = 0; \qquad \frac{d^2T}{dt^2} - p^2 T$$

$$m^2 - pa^2 = 0 \qquad m^2 - a^2 p^2 = 0$$

$$m = \pm p \qquad m = \pm ap$$

$$X(x) = (c_5 e^{px} + c_6 e^{-px}) \qquad T(t) = (c_7 e^{pat} + c_8 e^{-pat})$$

Case 3. If k is negative then $k = -p^2$ (say)

$$X'' + p^2 X = 0 \qquad T'' + p^2 a^2 t = 0$$

$$\frac{d^2X}{dx^2} + p^2 X = 0; \qquad \frac{d^2T}{dt^2} + p^2 T$$

$$m^2 + p^2 = 0 \qquad m^2 + a^2 p^2 = 0$$

$$m = \pm pi \qquad m = \pm api$$

$$X(x) = (c_9 \cos px + c_{10} \sin px) \quad T(t) = (c_{11} \cos pat + c_{12} \sin pat)$$

Solution of wave equation by separation of variables

Hence all possible solutions of the wave equation are

$$y(x,t) = (c_1 x + c_2)(c_3 t + c_4) \tag{5}$$

$$y(x,t) = (c_5 e^{px} + c_6 e^{-px})(c_7 e^{pat} + c_8 e^{-pat}) \tag{6}$$

$$y(x,t) = (c_9 \cos px + c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat) \tag{7}$$

Out of these three solution we have to choose the correct solution which is suitable for the given boundary conditions and physical nature of the problem. For the vibration of the string problem, the two boundary conditions $y(0, t) = 0$ and $y(l, t) = 0$ is always possible since the two ends at $x = 0$ and $x = l$ are fixed. Now we applying these two boundary conditions to find out the correct solution.
Applying the condition $y(0, t) = 0$ for the equation (5) we get

$$\begin{aligned} y(x,t) =&(c_1 x + c_2)(c_3 t + c_4) \\ y(0,t) =&c_2(c_3 t + c_4) = 0 \\ &\implies c_2 = 0 \end{aligned}$$

Applying the condition $y(l, t) = 0$, we get

$$\begin{aligned} y(l,t) =&c_1(l)(c_3 t + c_4) = 0 \\ &l \neq 0 \quad \text{and} \quad (c_3 t + c_4) \neq 0 \\ &\implies c_1 = 0 \end{aligned}$$

Solution of wave equation by separation of variables

Hence we get $y(x,t)=0$ as trivial solution. Applying the condition $y(0,t)=0$ for the equation (6) we get

$$
\begin{aligned}
y(0,t) = &(c_5+c_6)(c_7e^{pat}+c_8e^{-pat})=0\\
&(c_7e^{pat}+c_8e^{-pat})\neq 0\\
&\implies (c_5+c_6)=0
\end{aligned}
$$

Applying the condition $y(l,t)=0$, we get

$$
\begin{aligned}
y(l,t) = &(c_5e^{pl}+c_6e^{-pl})(c_7e^{pat}+c_8e^{-pat})=0\\
&l\neq 0 \quad \text{and} \quad (c_7e^{pat}+c_8e^{-pat})\neq 0\\
&\implies (c_5e^{pl}+c_6e^{-pl})=0\\
&c_5=0, c_6=0
\end{aligned}
$$

Solution of wave equation by separation of variables

In this case also we get $y(x,t) = 0$ as trivial solution.
Hence the correct solution is

$$y(x,t) = (c_9 \cos px + c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat).$$

Example 9.

A tightly stretched flexible string has its ends fixed at $x = 0$ *and* $x = \ell$. *At the time* $t = 0$, *the string is given a shape defined by*

$$F(x) = kx^2(\ell - x),$$

where k *is a constant, and then released from rest. Find the displacement at any point* x *of the string at any time* $t > 0$.

Solution of a problem on vibrating string with initial velocity zero

The partial differential equation corresponding to the BVP is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

The boundary conditions are

$$\begin{aligned}
(i)\, y(0,t) &= 0,\ \forall\, t > 0\\
(ii)\, y(l,t) &= 0,\ \forall\, t > 0\\
(iii)\ \left(\frac{\partial y}{\partial t}\right)_{t=0} &= 0,\ 0 < x < l\\
(iv)\, y(x,0) &= k(lx^2 - x^3),\ 0 < x < l
\end{aligned}$$

Solution of a problem on vibrating string with initial velocity zero

Solving (1) by method of separation of variable we get,

$$y(x,t) = (c_1 x + c_2)(c_3 t + c_4) \tag{2}$$
$$y(x,t) = (c_5 e^{px} + c_6 e^{-px})(c_7 e^{pat} + c_8 e^{-pat}) \tag{3}$$
$$y(x,t) = (c_9 \cos px + c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat) \tag{4}$$

Using the boundary conditions (i) and (ii) both the equation (2) and (3) are trivial solutions, that is $y(x,t) = 0$
Hence the most suitable solution for equation (1) is

$$y(x,t) = (c_9 \cos px + c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat)$$

Solution of a problem on vibrating string with initial velocity zero

Using the boundary condition (i), we get

$$c_9(c_{11} \cos pat + c_{12} \sin pat) = 0$$

But $(c_{11} \cos pat + c_{12} \sin pat) \neq 0$

$$\implies c_9 = 0$$

$$y(x, t) = (c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat) \tag{5}$$

Solution of a problem on vibrating string with initial velocity zero

Using the boundary condition (ii), we get

$$c_{10} \sin pl (c_{11} \cos pat + c_{12} \sin pat) = 0$$

But $(c_{11} \cos pat + c_{12} \sin pat) \neq 0$ and $c_{10} \neq 0$ [if $c_{10} = 0$, we get a trivial solution as $y(x,t) = 0$]

$$\begin{aligned} \implies \sin pl &= 0 \\ pl &= n\pi \\ p &= \frac{n\pi}{l} \end{aligned}$$

Hence equation (5) becomes

$$y(x,t) = \left(c_{10} \sin \frac{n\pi x}{l}\right)\left(c_{11} \cos \frac{n\pi at}{l} + c_{12} \sin \frac{n\pi at}{l}\right) \tag{6}$$

Solution of a problem on vibrating string with initial velocity zero

To use the boundary conditions (iii), first differentiating (6) with respect to t we get

$$\frac{\partial y(x,t)}{\partial t} = \left(c_{10}\sin\frac{n\pi x}{l}\right)\left(-c_{11}\frac{n\pi a}{l}\sin\frac{n\pi at}{l} + \frac{n\pi a}{l}c_{12}\cos\frac{n\pi at}{l}\right)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \Longrightarrow$$

$$\left(c_{10}\sin\frac{n\pi x}{l}\right)\left(0 + \frac{n\pi a}{l}c_{12}(1)\right) = 0$$

Solution of a problem on vibrating string with initial velocity zero

But $c_{10} \neq 0, \sin \frac{n\pi x}{l} \neq 0$ and $\frac{n\pi x}{l} \neq 0$ and hence $c_{12} = 0$, therefore

$$y(x,t) = c_{10}c_{11} \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \tag{7}$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \tag{8}$$

Solution of a problem on vibrating string with initial velocity zero

Using the boundary condition (iv) in (8) we get

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} = kx^2(l-x)$$

where c_n is given by

$$c_n = \frac{2}{l}\int_0^l k(lx^2 - x^3)\sin\frac{n\pi x}{l}dx$$

Solution of a problem on vibrating string with initial velocity zero

$$
\begin{aligned}
c_n &= \frac{2k}{l}\left[(lx^2 - x^3)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (2lx - 3x^2)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (2l - 6x)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right) - (-6)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n^4\pi^4}{l^4}}\right)\right]_0^l \\
&= \frac{2k}{l}\left[0 + 0 + \frac{l^3}{n^3\pi^3}(-4l\cos n\pi - 2l\cos 0) + 0\right] \\
&= \frac{2k}{l}\,\frac{l^3}{n^3\pi^3}[(-2l)(2(-1)^n + 1)] \\
&= -\frac{4kl^3}{n^3\pi^3}[1 + 2(-1)^n]
\end{aligned}
$$

$$
y(x,t) = \sum_{n=1}^{\infty} -\frac{4kl^3}{n^3\pi^3}[1 + 2(-1)^n]\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}
$$

Solution of a problem on vibrating string with initial velocity zero

Problem on vibrating string with initial velocity zero

Example 10.

A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At the time t=0, the string is given by the displacement

$$f(x) = kx(l - x),$$

where k is a constant, and then released from rest. Find the displacement at any point x of the string at any time $t > 0$.

Solution of a problem on vibrating string with initial velocity zero

The Partial differential equation corresponding to the BVP is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

The boundary conditions are

$$\begin{aligned}
(i)\, y(0,t) &= 0, \forall\, t > 0 \\
(ii)\, y(l,t) &= 0, \forall\, t > 0 \\
(iii)\ \left(\frac{\partial y}{\partial t}\right)_{t=0} &= 0,\ 0 < x < l \\
(iv)\, y(x,0) &= k(lx - x^2),\ 0 < x < l
\end{aligned}$$

Solution of a problem on vibrating string with initial velocity zero

Solving (1) by method of separation of variable we get,

$$y(x,t) = (c_1 x + c_2)(c_3 t + c_4) \tag{2}$$

$$y(x,t) = (c_5 e^{px} + c_6 e^{-px})(c_7 e^{pat} + c_8 e^{-pat}) \tag{3}$$

$$y(x,t) = (c_9 \cos px + c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat) \tag{4}$$

Using the boundary conditions (i) and (ii) both the equation (2) and (3) are trivial solutions, that is $y(x,t) = 0$
Hence the most suitable solution for equation (1) is

$$y(x,t) = (c_9 \cos px + c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat)$$

Solution of a problem on vibrating string with initial velocity zero

Using the boundary condition (i), we get

$$c_9(c_{11}\cos pat + c_{12}\sin pat) = 0$$

But $(c_{11}\cos pat + c_{12}\sin pat) \neq 0$

$$\implies c_9 = 0$$

$$y(x,t) = (c_{10}\sin px)(c_{11}\cos pat + c_{12}\sin pat) \tag{5}$$

Using the boundary condition (ii), we get

$$c_{10}\sin pl(c_{11}\cos pat + c_{12}\sin pat) = 0$$

Solution of a problem on vibrating string with initial velocity zero

But $(c_{11} \cos pat + c_{12} \sin pat) \neq 0$ and $c_{10} \neq 0$ [if $c_{10} = 0$, we get a trivial solution as $y(x,t) = 0$]

$$\begin{aligned} \sin pl &= 0 \\ pl &= n\pi \\ p &= \frac{n\pi}{l} \end{aligned}$$

Hence equation (5) becomes

$$y(x,t) = \left(c_{10} \sin \frac{n\pi x}{l}\right)\left(c_{11} \cos \frac{n\pi at}{l} + c_{12} \sin \frac{n\pi at}{l}\right) \tag{6}$$

Solution of a problem on vibrating string with initial velocity zero

To use the boundary conditions (iii), first differentiating (6) with respect to t we get

$$\frac{\partial y(x,t)}{\partial t} = \left(c_{10}\sin\frac{n\pi x}{l}\right)\left(-c_{11}\frac{n\pi a}{l}\sin\frac{n\pi at}{l} + \frac{n\pi a}{l}c_{12}\cos\frac{n\pi at}{l}\right)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \Longrightarrow$$

$$\left(c_{10}\sin\frac{n\pi x}{l}\right)\left(0 + \frac{n\pi a}{l}c_{12}(1)\right) = 0$$

Solution of a problem on vibrating string with initial velocity zero

But $c_{10} \neq 0, \sin \frac{n\pi x}{l} \neq 0$ and $\frac{n\pi x}{l} \neq 0$ and hence $c_{12} = 0$, therefore

$$y(x,t) = c_{10} c_{11} \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l} \tag{7}$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l} \tag{8}$$

Solution of a problem on vibrating string with initial velocity zero

Using the boundary condition (iv) in (8) we get

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} = kx(l-x)$$

where c_n is given by

$$\begin{aligned}
c_n &= \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi x}{l}\,dx \\
&= \frac{2k}{l}\left[(lx-x^2)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l \\
&= \frac{2k}{l}\left[0 + 0 - \frac{2l^3}{n^3\pi^3}(\cos n\pi - \cos 0)\right] \\
&= \frac{4kl^2}{n^3\pi^3}[1-(-1)^n]
\end{aligned}$$

$$y(x,t) = \sum_{n=1}^{\infty} \frac{4kl^2}{n^3\pi^3}[1-(-1)^n]\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}.$$

Problem on vibrating string with initial velocity zero

Example 11.

A tightly stretched string of length l has its ends fastened at $x = 0$, $x = l$. The mid point of the string is then taken to height h and the released from rest from that position. Find the lateral displacement of a point of the string at time t from the instant of release.

Solution of a problem on vibrating string with initial velocity zero

The partial differential equation corresponding to the BVP is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

Equation of the line OA is

$$y - 0 = \frac{h-0}{\frac{l}{2}-0}(x-0)$$
$$y = \frac{2h}{l}x$$

Equation of the line AB is

$$y - 0 = \frac{h-0}{\frac{l}{2}-l}(x-l)$$
$$y = \frac{2h}{l}(l-x)$$

The boundary conditions are

Solution of a problem on vibrating string with initial velocity zero

Solving (1) by method of separation of variable we get,

$$y(x,t) = (c_1 x + c_2)(c_3 t + c_4) \tag{2}$$

$$y(x,t) = (c_5 e^{px} + c_6 e^{-px})(c_7 e^{pat} + c_8 e^{-pat}) \tag{3}$$

$$y(x,t) = (c_9 \cos px + c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat) \tag{4}$$

Using the boundary conditions (i) and (ii) both the equation (2) and (3) are trivial solutions, that is $y(x,t) = 0$

Hence the most suitable solution for equation (1) is

$$y(x,t) = (c_9 \cos px + c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat)$$

Solution of a problem on vibrating string with initial velocity zero

Using the boundary condition (i), we get

$$c_9(c_{11}\cos pat + c_{12}\sin pat) = 0$$

But $(c_{11}\cos pat + c_{12}\sin pat) \neq 0$

$$\implies c_9 = 0$$

$$y(x,t) = (c_{10}\sin px)(c_{11}\cos pat + c_{12}\sin pat) \tag{5}$$

Using the boundary condition (ii), we get

$$c_{10}\sin pl(c_{11}\cos pat + c_{12}\sin pat) = 0$$

Solution of a problem on vibrating string with initial velocity zero

But $(c_{11} \cos pat + c_{12} \sin pat) \neq 0$ and $c_{10} \neq 0$ [if $c_{10} = 0$, we get a trivial solution as $y(x,t) = 0$]

$$\begin{aligned} \implies \sin pl &= 0 \\ pl &= n\pi \\ p &= \frac{n\pi}{l} \end{aligned}$$

Hence equation (5) becomes

$$y(x,t) = \left(c_{10} \sin \frac{n\pi x}{l}\right)\left(c_{11} \cos \frac{n\pi at}{l} + c_{12} \sin \frac{n\pi at}{l}\right) \tag{6}$$

Solution of a problem on vibrating string with initial velocity zero

To use the boundary conditions (iii), first differentiating (6) with respect to t we get

$$\frac{\partial y(x,t)}{\partial t} = \left(c_{10}\sin\frac{n\pi x}{l}\right)\left(-c_{11}\frac{n\pi a}{l}\sin\frac{n\pi at}{l} + \frac{n\pi a}{l}c_{12}\cos\frac{n\pi at}{l}\right)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \Longrightarrow$$

$$\left(c_{10}\sin\frac{n\pi x}{l}\right)\left(0 + \frac{n\pi a}{l}c_{12}(1)\right) = 0$$

Solution of a problem on vibrating string with initial velocity zero

But $c_{10} \neq 0, \sin\frac{n\pi x}{l} \neq 0$ and $\frac{n\pi x}{l} \neq 0$ and hence $c_{12} = 0$, therefore

$$y(x,t) = c_{10}c_{11}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \tag{7}$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \tag{8}$$

Using the boundary condition (iv) in (8) we get

$$\begin{aligned} y(x,0) &= \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \\ &= \begin{cases} \frac{2hx}{l} & \text{when} \quad 0 < x < \frac{l}{2} \\ \frac{2h}{l}(l-x) & \text{when} \quad \frac{l}{2} < x < l. \end{cases} \end{aligned}$$

Solution of a problem on vibrating string with initial velocity zero

where c_n is given by

$$\begin{aligned}
c_n &= \frac{2}{l}\left[\int_0^{\frac{l}{2}} \frac{2h}{l} x \sin\frac{n\pi x}{l}\,dx + \int_{\frac{l}{2}}^{l} \frac{2h}{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right] \\
&= \frac{4h}{l^2}\left[x\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{\frac{l}{2}} + \frac{4h}{l^2}\left[(l-x)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{\frac{l}{2}}^{l} \\
&= \frac{4h}{l^2}\left[\frac{l}{2}\frac{l}{n\pi}\left(-\cos\frac{n\pi}{2} - 0\right) + \frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\right] + \frac{4h}{l^2}\left[0 - \frac{l}{2}\frac{l}{n\pi}\left(-\cos\frac{n\pi}{2}\right) - \frac{l^2}{n^2\pi^2}\left(0 - \sin\frac{n\pi}{2}\right)\right] \\
&= \frac{4h}{l^2}\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2} \\
&= \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}.
\end{aligned}$$

$$y(x,t) = \sum_{n=1}^{\infty} \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l}.$$

Problem on vibrating string with initial velocity zero

Example 12.

A tightly stretched string of length $2l$ has its ends fastened at $x = 0$, $x = 2l$. The mid point of the string is then taken to height b and the released from rest from that position. Find the lateral displacement of a point of the string at time t from the instant of release.

Solution of a problem on vibrating string with initial velocity zero

The partial differential equation corresponding to the BVP is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

Equation of the line OA is

$$y - 0 = \frac{b-0}{l-0}(x-0)$$
$$y = \frac{b}{l}x$$

Equation of the line AB is

$$y - 0 = \frac{b-0}{l-2l}(x-2l)$$
$$y = \frac{b}{l}(2l-x)$$

Solution of a problem on vibrating string with initial velocity zero

The boundary conditions are

$$
\begin{aligned}
(i)\, y(0,t) &= 0, \forall\, t > 0 \\
(ii)\, y(2l,t) &= 0, \forall\, t > 0 \\
(iii)\, \left(\frac{\partial y}{\partial t}\right)_{t=0} &= 0,\ 0 < x < 2l \\
(iv)\, y(x,0) &= \begin{cases} \frac{bx}{l} & \text{when} \quad 0 < x < l \\ \frac{b}{l}(2l - x) & \text{when} \quad l < x < 2l. \end{cases}
\end{aligned}
$$

Solution of a problem on vibrating string with initial velocity zero

Solving (1) by method of separation of variable we get,

$$y(x,t) = (c_1 x + c_2)(c_3 t + c_4) \tag{2}$$

$$y(x,t) = (c_5 e^{px} + c_6 e^{-px})(c_7 e^{pat} + c_8 e^{-pat}) \tag{3}$$

$$y(x,t) = (c_9 \cos px + c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat) \tag{4}$$

Using the boundary conditions (i) and (ii) both the equation (2) and (3) are trivial solutions, that is $y(x,t) = 0$
Hence the most suitable solution for equation (1) is

$$y(x,t) = (c_9 \cos px + c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat)$$

Using the boundary condition (i), we get

$$c_9(c_{11} \cos pat + c_{12} \sin pat) = 0.$$

Solution of a problem on vibrating string with initial velocity zero

But $(c_{11} \cos pat + c_{12} \sin pat) \neq 0$

$$\implies c_9 = 0$$

$$y(x,t) = (c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat) \tag{5}$$

Using the boundary condition (ii), we get

$$c_{10} \sin p2l(c_{11} \cos pat + c_{12} \sin pat) = 0$$

But $(c_{11} \cos pat + c_{12} \sin pat) \neq 0$ and $c_{10} \neq 0$ [if $c_{10} = 0$, we get a trivial solution as $y(x,t) = 0$]

$$\begin{aligned} \implies \sin p2l &= 0 \\ p2l &= n\pi \\ p &= \frac{n\pi}{2l}. \end{aligned}$$

Solution of a problem on vibrating string with initial velocity zero

Hence equation (5) becomes

$$y(x,t) = \left(c_{10}\sin\frac{n\pi x}{2l}\right)\left(c_{11}\cos\frac{n\pi at}{2l} + c_{12}\sin\frac{n\pi at}{2l}\right) \qquad (6)$$

To use the boundary conditions (iii), first differentiating (6) with respect to t we get

$$\frac{\partial y(x,t)}{\partial t} = \left(c_{10}\sin\frac{n\pi x}{2l}\right)\left(-c_{11}\frac{n\pi a}{2l}\sin\frac{n\pi at}{2l} + \frac{n\pi a}{2l}c_{12}\cos\frac{n\pi at}{2l}\right)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \Longrightarrow$$

$$\left(c_{10}\sin\frac{n\pi x}{2l}\right)\left(0 + \frac{n\pi a}{2l}c_{12}(1)\right) = 0$$

Solution of a problem on vibrating string with initial velocity zero

But $c_{10} \neq 0, \sin \frac{n\pi x}{l} \neq 0$ and $\frac{n\pi x}{2l} \neq 0$ and hence $c_{12} = 0$, therefore

$$y(x,t) = c_{10}c_{11} \sin \frac{n\pi x}{2l} \cos \frac{n\pi at}{2l} \tag{7}$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{2l} \cos \frac{n\pi at}{2l} \tag{8}$$

Solution of a problem on vibrating string with initial velocity zero

Using the boundary condition (iv) in (8) we get

$$y(x,0)=\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{2l} = \begin{cases} \frac{bx}{l} & \text{when} \quad 0<x<l \\ \frac{b}{l}(2l-x) & \text{when} \quad l<x<2l. \end{cases}$$

where c_n is given by

$$\begin{aligned} c_n &= \frac{2}{2l}\left[\int_0^l \frac{b}{l}x\sin\frac{n\pi x}{2l}\,dx + \int_l^{2l}\frac{b}{l}(2l-x)\sin\frac{n\pi x}{2l}\,dx\right] \\ &= \frac{b}{l^2}\left[x\left(-\frac{\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - 1\left(-\frac{\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right)\right]_0^l + \frac{b}{l^2}\left[(2l-x)\left(-\frac{\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (-1)\left(-\frac{\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right)\right]_l^{2l} \\ &= \frac{b}{l^2}\left[(l)\frac{2l}{n\pi}\left(-\cos\frac{n\pi}{2}-0\right) + \frac{4l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\right] + \frac{b}{2l^2}\left[0-(l)\frac{2l}{n\pi}\left(-\cos\frac{n\pi}{2}\right) - \frac{4l^2}{n^2\pi^2}\left(0-\sin\frac{n\pi}{2}\right)\right] \\ &= \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}. \end{aligned}$$

$$y(x,t)=\sum_{n=1}^{\infty}\frac{4b}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\cos\frac{n\pi at}{2l}.$$

Problem on vibrating string with initial velocity zero

Example 13.

A tightly stretched string with fixed end points $x = 0$ *and* $x = l$ *is initially in a position given by* $y(x, 0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. *It is released from rest from this position. Find the displacement at any time.*

Problem on vibrating string with initial velocity zero

The partial differential equation corresponding to the BVP is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

The boundary conditions are

$$\begin{aligned}
(i)\, y(0,t) &= 0, \forall\, t > 0 \\
(ii)\, y(l,t) &= 0, \forall\, t > 0 \\
(iii)\ \left(\frac{\partial y}{\partial t}\right)_{t=0} &= 0,\ 0 < x < l \\
(iv)\, y(x,0) &= y_0 \sin^3\left(\frac{\pi x}{l}\right),\ 0 < x < l
\end{aligned}$$

Solution of a problem on vibrating string with initial velocity zero

Solving (1) by method of separation of variable we get,

$$y(x,t) = (c_1 x + c_2)(c_3 t + c_4) \tag{2}$$
$$y(x,t) = (c_5 e^{px} + c_6 e^{-px})(c_7 e^{pat} + c_8 e^{-pat}) \tag{3}$$
$$y(x,t) = (c_9 \cos px + c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat) \tag{4}$$

Using the boundary conditions (i) and (ii) both the equation (2) and (3) are trivial solutions, that is $y(x,t) = 0$
Hence the most suitable solution for equation (1) is

$$y(x,t) = (c_9 \cos px + c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat)$$

Solution of a problem on vibrating string with initial velocity zero

Using the boundary condition (i), we get

$$c_9(c_{11} \cos pat + c_{12} \sin pat) = 0$$

But $(c_{11} \cos pat + c_{12} \sin pat) \neq 0$

$$\implies c_9 = 0$$

$$y(x,t) = (c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat) \tag{5}$$

Using the boundary condition (ii), we get

$$c_{10} \sin pl(c_{11} \cos pat + c_{12} \sin pat) = 0$$

Solution of a problem on vibrating string with initial velocity zero

But $(c_{11} \cos pat + c_{12} \sin pat) \neq 0$ and $c_{10} \neq 0$ [if $c_{10} = 0$, we get a trivial solution as $y(x, t) = 0$]

$$\begin{aligned} \sin pl &= 0 \\ pl &= n\pi \\ p &= \frac{n\pi}{l} \end{aligned}$$

Hence equation (5) becomes

$$y(x,t) = \left(c_{10} \sin \frac{n\pi x}{l}\right) \left(c_{11} \cos \frac{n\pi at}{l} + c_{12} \sin \frac{n\pi at}{l}\right) \tag{6}$$

Solution of a problem on vibrating string with initial velocity zero

To use the boundary conditions (iii), first differentiating (6) with respect to t we get

$$\frac{\partial y(x,t)}{\partial t} = \left(c_{10} \sin \frac{n\pi x}{l}\right) \left(-c_{11} \frac{n\pi a}{l} \sin \frac{n\pi at}{l} + \frac{n\pi a}{l} c_{12} \cos \frac{n\pi at}{l}\right)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \Longrightarrow$$

$$\left(c_{10} \sin \frac{n\pi x}{l}\right) \left(0 + \frac{n\pi a}{l} c_{12}(1)\right) = 0$$

But $c_{10} \neq 0, \sin \frac{n\pi x}{l} \neq 0$ and $\frac{n\pi x}{l} \neq 0$ and hence $c_{12} = 0$, therefore

$$y(x,t) = c_{10} c_{11} \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \tag{7}$$

Solution of a problem on vibrating string with initial velocity zero

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \tag{8}$$

Using the boundary condition (iv) in (8) we get

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = y_0 \sin^3\left(\frac{x\pi}{l}\right)$$

Hence

$$\begin{aligned}
\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} &= y_0 \sin^3\left(\frac{x\pi}{l}\right) \\
&= \frac{y_0}{4}\left(3\sin\frac{x\pi}{l} - \sin\frac{3x\pi}{l}\right) c_1 \sin\frac{x\pi}{l} + c_2 \sin\frac{2x\pi}{l} + c_3 \sin\frac{3x\pi}{l} + \cdots \\
&= \frac{y_0}{4}\left(3\sin\frac{x\pi}{l} - \sin\frac{3x\pi}{l}\right)
\end{aligned}$$

Solution of a problem on vibrating string with initial velocity zero

Equating the coefficient on both side we get
$c_1 = \frac{3y_0}{4}$, $c_3 = -\frac{y_0}{4}$ and $c_n = 0, \forall\, n \neq 1, 3$. Hence

$$y(x,t) = \frac{3y_0}{4} \sin \frac{x\pi}{l} \cos \frac{\pi a t}{l} - \frac{y_0}{4} \sin \frac{3x\pi}{l} \cos \frac{3\pi a t}{l}$$

Problem on vibrating string with initial velocity zero

Example 14.

A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y(x, 0) = y_0 \sin\left(\frac{\pi x}{l}\right)$. It is released from rest from this position. Find the displacement at any time.

Solution of a problem on vibrating string with initial velocity zero

The partial differential equation corresponding to the BVP is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

The boundary conditions are

$$\begin{aligned}
(i)\; y(0,t) &= 0, \forall\, t > 0 \\
(ii)\; y(l,t) &= 0, \forall\, t > 0 \\
(iii)\; \left(\frac{\partial y}{\partial t}\right)_{t=0} &= 0,\; 0 < x < l \\
(iv)\; y(x,0) &= y_0 \sin\left(\frac{\pi x}{l}\right),\; 0 < x < l
\end{aligned}$$

Solution of a problem on vibrating string with initial velocity zero

Solving (1) by method of separation of variable we get,

$$y(x,t) = (c_1 x + c_2)(c_3 t + c_4) \tag{2}$$

$$y(x,t) = (c_5 e^{px} + c_6 e^{-px})(c_7 e^{pat} + c_8 e^{-pat}) \tag{3}$$

$$y(x,t) = (c_9 \cos px + c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat) \tag{4}$$

Solution of a problem on vibrating string with initial velocity zero

Using the boundary conditions (i) and (ii) both the equation (2) and (3) are trivial solutions, that is $y(x,t) = 0$
Hence the most suitable solution for equation (1) is

$$y(x,t) = (c_9 \cos px + c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat)$$

Using the boundary condition (i), we get

$$c_9(c_{11} \cos pat + c_{12} \sin pat) = 0$$

But $(c_{11} \cos pat + c_{12} \sin pat) \neq 0$

$$\implies c_9 = 0$$

$$y(x,t) = (c_{10} \sin px)(c_{11} \cos pat + c_{12} \sin pat) \tag{5}$$

Solution of a problem on vibrating string with initial velocity zero

Using the boundary condition (ii), we get

$$c_{10} \sin pl (c_{11} \cos pat + c_{12} \sin pat) = 0$$

But $(c_{11} \cos pat + c_{12} \sin pat) \neq 0$ and $c_{10} \neq 0$ [if $c_{10} = 0$, we get a trivial solution as $y(x, t) = 0$]

$$\begin{aligned} \implies \sin pl &= 0 \\ pl &= n\pi \\ p &= \frac{n\pi}{l} \end{aligned}$$

Hence equation (5) becomes

$$y(x,t) = \left(c_{10} \sin \frac{n\pi x}{l}\right)\left(c_{11} \cos \frac{n\pi at}{l} + c_{12} \sin \frac{n\pi at}{l}\right) \tag{6}$$

Solution of a problem on vibrating string with initial velocity zero

To use the boundary conditions (iii), first differentiating (6) with respect to t we get

$$\frac{\partial y(x,t)}{\partial t} = \left(c_{10}\sin\frac{n\pi x}{l}\right)\left(-c_{11}\frac{n\pi a}{l}\sin\frac{n\pi at}{l} + \frac{n\pi a}{l}c_{12}\cos\frac{n\pi at}{l}\right)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \Longrightarrow$$

$$\left(c_{10}\sin\frac{n\pi x}{l}\right)\left(0 + \frac{n\pi a}{l}c_{12}(1)\right) = 0$$

Solution of a problem on vibrating string with initial velocity zero

But $c_{10} \neq 0, \sin\frac{n\pi x}{l} \neq 0$ and $\frac{n\pi x}{l} \neq 0$ and hence $c_{12} = 0$, therefore

$$y(x,t) = c_{10}c_{11} \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \tag{7}$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \tag{8}$$

Using the boundary condition (iv) in (8) we get

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = y_0 \sin\left(\frac{x\pi}{l}\right)$$

Solution of a problem on vibrating string with initial velocity zero

Hence

$$\sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} = y_0 \sin\left(\frac{x\pi}{l}\right)$$

$$c_1 \sin \frac{x\pi}{l} + c_2 \sin \frac{2x\pi}{l} + c_3 \sin \frac{3x\pi}{l} + \cdots = y_0 \sin\left(\frac{x\pi}{l}\right)$$

Equating the coefficient on both side we get
$c_1 = y_0$, and $c_n = 0 \ \forall n = 2, 3, \ldots$. Hence

$$y(x,t) = y_0 \sin \frac{x\pi}{l} \cos \frac{\pi a t}{l}$$

Solution of a problem on vibrating string with initial velocity zero

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